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Fixed Point Results in Hilbert Space via Altering Distance Functions

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Abstract

Drawing on the principles of functional analysis within the framework of Hilbert space, this research aims to establish fixed point results for distinct mappings under newly proposed rational contractions employing altering distance functions. The study employs specific mathematical techniques, including Banach Contraction Principle, iterative methods, and relational mappings that incorporate partial orders and metric spaces. In particular, we construct appropriate complete metric spaces and apply rational expressions to define contraction conditions. The methods also involve proving convergence through iterative sequences and leveraging the properties of partial orders to manage relational structures. Additionally, the findings are illustrated with examples to enhance understanding. The paper further explores the derivation of corollaries, which serve as specialized instances of the main results. Through iterative processes and methodological rigor, this work advances understanding in the field, providing valuable insights into the properties of fixed points under the specified rational contractions.

Keywords: Hilbert space; inner product space, normed linear space; parallelogram law.

1 Introduction

In the field of non-linear analysis, fixed point theory plays a significant role. In 1911, Brouwer first described about fixed point results in Euclidean space but it was not a constructive result. In 1922, a famous mathematician S. Banach [2] established his famous Banach contraction principle and showed fixed point result using this contraction. After that, Kannan [13, 14], Chatterjee [7] and many more mathematicians [9, 17] worked on this field and presented their respective results. Subsequent research has expanded these principles to various contexts, including metric and ordered metric spaces [15, 31]. A revolution in contraction mapping came in 2004 when Berinde [3] introduced the idea of weak contraction and using this he showed fixed point results. There subsists different types of metric spaces in which inner product space is one of them. When the inner product space attains completeness property, it can be known as Hilbert space as named by famous mathematician David Hilbert.

Hilbert space is a vector space that is equipped with inner product space defining distance function that's why this space can be a complete metric space. In 1965, Browder [4] demonstrated certain outcomes regarding fixed points concerning mappings that are noncompact within the context of Hilbert space. After a couple of years, Browder and Petryshyn [5] presented another work on the construction of fixed points of non-linear mappings in Hilbert space. In 1966, Petryshyn [26] alone worked on the construction of fixed points of demicompact mappings in Hilbert space. A few years later, in 1975, Baillon [1] presented some fixed point results in Hilbert space using non-linear contractions. In 1991, Koparde et al. [16] presented their results in Hilbert space using Kannan type mappings. In 2005, Debnath et al. [10] worked on Hilbert space and its applications published as a book and established several convergence related results concerning fixed points for Lipschitz pseudo-contractions within the realm of Hilbert space. Qin et al. [28, 27] derived notable findings pertaining to equilibrium problems and fixed point problems within the domain of Hilbert space. Recent investigations have provided comprehensive guides and new insights into fixed point analysis and applications. Ozer [20] worked in special metric space with triple fixed points and performed fixed point analysis. Ullah et al. [37] worked on convergence results using (k^*) iteration process in busemann spaces.

Additionally, studies on specific metric spaces and coupled fixed points in C*-algebra valued metric spaces have further enriched the literature. Omran [18] and Ozer [24] provided existance and uniqeness of coupled fixed point in C* algebra valued metric spaces. Ozer [21, 23] demonstrated existance and uniqeness of coupled fixed point in C* algebra valued b-metric spaces. Ozer [22] demonstrated existance and uniqeness of coupled fixed point in C* algebra valued g metric spaces. Recent works exploring special metric spaces with triple fixed points and boundary value problems of Caputo fractional differential equations provide valuable methodologies for our study [30].

In recent times several scientists presented their results on Hilbert space and other generalizations of metric spaces and produced applications in different fields. Berinde [3] gave the approximation of fixed points of weak contractions using the picard iteration. Cegielski [6] introduced iterative methods for fixed point problems in Hilbert spaces and Cholamjiak et al. [8] introduced a new concept of hybrid multivalued mappings in Hilbert spaces. Osilike et al. [19] gave weak and strong convergence theorems in real Hilbert spaces for a new class of nonspreading-type mappings more general than the class. Patel [25] worked on fixed point theorem in Hilbert space using weak and strong convergence. Some result on fixed point theorem in Hilbert space has been establised in [29]. In [32], $(\alpha\eta)$-contractive and $(\beta\chi)$-contractive mapping based fixed point theorems in fuzzy bipolar metric spaces and application to nonlinear Volterra integral equations. In [33], provided fixed-point results for mappings satisfying implicit relation in orthogonal fuzzy metric spaces. Ghosh et al. [11] introduced neutrosophic fuzzy metric space and its topological properties. Suantai et al. [34] worked on solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces. Takahashi et al. [35] gave fixed point theorems for new generalized hybrid mappings in Hilbert spaces and applications.

Tian [36] provide weak convergence theorem for zero points of inverse strongly monotone mapping and fixed point of non expansive mapping in Hilbert space. Ulucay et al. [38] introduced the concept of soft expert metric spaces. Wang et al. [39] gave viscosity extragradient method for an equilibrium problem and fixed point problem in Hilbert space. Zhou [40] provided convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces.

In the present study, we establish some fixed point results for distinct mappings under newly proposed rational contractions incorporating the usage of altering distance functions within the framework of Hilbert spaces. Additionally, the findings are illustrated with some examples to support the proven results. The paper further explores the derivation of consequenses, which serve as special cases of the presented main results.

2 Preliminaries

Definition 2.1. [12] *Consider $\mathscr{H}$ as a linear space. Then, $\mathscr{H}$ is known as a normed linear space if to each member $\check{\mathfrak{w}}$ of $\mathscr{H}$, there is attached an idiosyncratic real number, called the norm of this element and is prevailed by $||\check{\mathfrak{w}}||$ such that the follwing conditions are accomplished:*

(*i*) $||\check{\mathfrak{w}}|| \geq 0$ *and* $||\check{\mathfrak{w}}|| = 0$ *iff* $\check{\mathfrak{w}} = \theta$,

(*ii*) $||\check{\mathfrak{w}} + \check{\mathfrak{r}}|| \leq ||\check{\mathfrak{w}}|| + ||\check{\mathfrak{r}}||$,

(*iii*) $||\lambda\check{\mathfrak{w}}|| = |\lambda|\ ||\check{\mathfrak{w}}||$, λ *is a scalar,*

for all $\check{\mathfrak{w}}, \check{\mathfrak{r}} \in \mathscr{H}$. *This is denoted as* $(\mathscr{H}, ||.||)$.

Norm generates a speacial metric on a linear space. Given a normed linear space $(\mathscr{H}, ||.||)$, a function $\mathfrak{d} : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ defined as $\mathfrak{d}(\check{\mathfrak{w}}, \check{\mathfrak{r}}) = ||\check{\mathfrak{w}} - \check{\mathfrak{r}}||$ is a metric on $\mathscr{H}$ and is called metric generated by the norm. Thus, the function $\mathfrak{d}$ satisfies all the metric axioms:

(i) $||\check{\mathfrak{w}} - \check{\mathfrak{r}}|| \geq 0$,

(ii) $||\check{\mathfrak{w}} - \check{\mathfrak{r}}|| = 0$ iff $\check{\mathfrak{w}} = \check{\mathfrak{r}}$,

(iii) $||\check{\mathfrak{w}} - \check{\mathfrak{r}}|| = ||\check{\mathfrak{r}} - \check{\mathfrak{w}}||$,

(iv) $||\check{\mathfrak{w}} - \check{\mathfrak{k}}|| \leq ||\check{\mathfrak{w}} - \check{\mathfrak{r}}|| + ||\check{\mathfrak{r}} - \check{\mathfrak{k}}||$,

for all $\check{\mathfrak{w}}, \check{\mathfrak{r}}, \check{\mathfrak{k}} \in \mathscr{H}$.

Definition 2.2. *Given that $\mathscr{H}$ is a linear space defined over the scalar field $\mathfrak{C}$ of complex numbers, it qualifies as an inner product space if, for any pair of elements $\mathfrak{p}$ and $\mathfrak{q}$ belonging to $\mathscr{H}$, there exists a complex number denoted by $(\mathfrak{p}, \mathfrak{q})$, which serves as the scalar product or inner product of $\mathfrak{p}$ and $\mathfrak{q}$.*

If $\mathfrak{w}, \mathfrak{r}, \mathfrak{j} \in \mathscr{H}$ and μ is a scalar, then the inner product needs to satisfy the following axioms:

(i) $(\mathfrak{w}, \mathfrak{r}) = (\overline{\mathfrak{r}, \mathfrak{w}})$, where $(\overline{\mathfrak{r}, \mathfrak{w}})$ denotes the complex conjugate.

(ii) $(\mathfrak{w} + \mathfrak{j}, \mathfrak{r}) = (\mathfrak{w}, \mathfrak{r}) + (\mathfrak{j}, \mathfrak{r})$.

(iii) $(\mu\mathfrak{w}, \mathfrak{r}) = \mu(\mathfrak{w}, \mathfrak{r})$.

(iv) $(\mathfrak{w}, \mathfrak{r}) \geq 0$ and $(\mathfrak{w}; \mathfrak{r}) = 0$ iff $\mathfrak{w} = 0$.

Definition 2.3. [5] *When an inner product space $\mathscr{H}$ becomes complete, then it is known as Hilbert Space.*

Definition 2.4. (***Parallelogram Law***)*: Suppose $\mathscr{H}$ is an inner product space, then,*

$$||\mathfrak{w} + \mathfrak{r}||^2 + ||\mathfrak{w} - \mathfrak{r}||^2 = 2||\mathfrak{w}||^2 + 2||\mathfrak{r}||^2,$$

for all $\mathfrak{w}, \mathfrak{r} \in \mathscr{H}$.

A Banach space is said to be a Hilbert space iff parallelogram law holds.

3 Main Results

In this section, firstly we present some definitions which will be useful for establishing the main results:

Definition 3.1. *Consider a function $\chi : \mathscr{H} \times \mathscr{H} \to [0, \infty)$. Then, $\mathscr{S} : \mathscr{H} \to \mathscr{H}$ is purported to be χ admissible if $\chi(\mathfrak{w}_0, \mathfrak{j}_0) \geq 1$ and $||\mathfrak{w}_1 - \mathscr{S}(\mathfrak{w}_0)|| = ||\mathfrak{j}_1 - \mathscr{S}(\mathfrak{j}_0)|| = 0$ implies $\chi(\mathfrak{w}_1, \mathfrak{j}_1) \geq 1$, for all $\mathfrak{w}_1, \mathfrak{j}_1, \mathfrak{w}_0, \mathfrak{j}_0 \in \mathscr{H}$, where $\mathscr{H}$ is a Hilbert space.*

Next, we present the definition of altering distance function which serves a basis for the main results.

Definition 3.2. *Consider Γ as an assembly of functions $\tau : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$ such that:*

(*i*) *τ must be increasing.*

(*ii*) *For each sequence $\{\mathfrak{w}_\mathbf{n}\} \in [0, \infty)$, $\lim_{\mathbf{n}\to\infty} \mathfrak{w}_\mathbf{n} = 0$ iff $\lim_{\mathbf{n}\to\infty} \tau(\mathfrak{w}_\mathbf{n}) = 0$.*

(*iii*) *τ must attain continuity.*

Definition 3.3. *Consider Σ, Ω as an assembly of functions $\eta, \zeta : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$ such that:*

(*i*) *η, ζ must be increasing.*

(*ii*) *Both must attain continuity.*

(*iii*) *$\eta(0) = 0$, $\eta(\mathfrak{f}) < \mathfrak{f}$ for each $\mathfrak{f} \in [0, \infty)$.*

Now, we present our main results.

Theorem 3.1. *Assume that $\mathscr{H}$ is a Hilbert space and $\mathscr{S} : \mathscr{H} \to \mathscr{H}$ be a continuous mapping such that for some $\mathfrak{a} \in (0, \infty)$ and there exist $\mathfrak{h}_1, \mathfrak{h}_2 \in (0, 1)$ for which $0 < \mathfrak{h}_1 + \mathfrak{h}_2 < 1$ satisfying,*

$$\chi(\ell, \mathfrak{s})\tau(||\mathscr{S}(\ell) - \mathscr{S}(\mathfrak{s})||^2) \leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell - \mathscr{S}(\ell)||^2 + \mathfrak{h}_2||\mathfrak{s} - \mathscr{S}(\mathfrak{s})||^2)\Big) - \frac{1}{\delta}\zeta\Big(\tau(||\ell - \mathscr{S}(\mathfrak{s})||^2 + ||\mathfrak{s} - \mathscr{S}(\ell)||^2)\Big),$$

for all $\delta \geq 2$ and $\ell, \mathfrak{s} \in \mathscr{H}$. Then, $\mathscr{S}$ has a unique invariant point in $\mathscr{H}$.

Proof. Suppose $\ell_1 = \mathscr{S}(\ell_0), \ell_2 = \mathscr{S}(\ell_1) = \mathscr{S}^2(\ell_0), \cdots, \ell_\mathfrak{n} = \mathscr{S}(\ell_{\mathfrak{n}-1}) = \mathscr{S}^\mathfrak{n}(\ell_0)$ for all $\mathfrak{n} \in \mathbb{N}$.

Again, let us assume $\xi(\ell_0, \ell_1) \geq 1$ and since $||\ell_1 - \mathscr{S}(\ell_0)|| = ||\ell_2 - \mathscr{S}(\ell_1)|| = 0$. Then, by the Definition 3.1 we can obtain, $\xi(\ell_1, \ell_2) \geq 1$.

By mathematical induction, we obtain $\xi(\ell_\mathfrak{n}, \ell_{\mathfrak{n}+1}) \geq 1$, for all $\mathfrak{n} \in \mathbb{N}$. Further we have

$$\begin{aligned}
\tau(||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}||^2) &= \tau\Big(||\mathscr{S}(\ell_{\mathfrak{n}-1}) - \mathscr{S}(\ell_\mathfrak{n})||^2\Big) \\
&\leq \chi(\ell_{\mathfrak{n}-1}, \ell_\mathfrak{n})\tau\Big(||\mathscr{S}(\ell_{\mathfrak{n}-1}) - \mathscr{S}(\ell_\mathfrak{n})||^2\Big) \\
&\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \mathscr{S}(\ell_{\mathfrak{n}-1})||^2 + \mathfrak{h}_2||\ell_\mathfrak{n} - \mathscr{S}(\ell_\mathfrak{n})||^2)\Big) \\
&\quad - \frac{1}{\delta}\zeta\Big(\tau(||\ell_{\mathfrak{n}-1} - \mathscr{S}(\ell_\mathfrak{n})||^2 + ||\ell_\mathfrak{n} - \mathscr{S}(\ell_{\mathfrak{n}-1})||^2)\Big) \\
&\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_\mathfrak{n}||^2 + \mathfrak{h}_2||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}||^2)\Big) \\
&< \eta\Big(\tau(\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_\mathfrak{n}||^2 + \mathfrak{h}_2||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}||^2)\Big) \\
&< \tau\Big(\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_\mathfrak{n}||^2 + \mathfrak{h}_2||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}||^2\Big) \\
\Rightarrow ||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}||^2 &< \mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_\mathfrak{n}||^2 + \mathfrak{h}_2||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}||^2 \\
\Rightarrow (1-\mathfrak{h}_2)||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}||^2 &< \mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_\mathfrak{n}||^2 \\
\Rightarrow ||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}||^2 &< \frac{\mathfrak{h}_1}{1-\mathfrak{h}_2}||\ell_{\mathfrak{n}-1} - \ell_\mathfrak{n}||^2 \\
&\vdots \\
&< \left(\frac{\mathfrak{h}_1}{1-\mathfrak{h}_2}\right)^\mathfrak{n}||\ell_0 - \ell_1||^2 \\
\Rightarrow ||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}|| &< \left(\frac{\mathfrak{h}_1}{1-\mathfrak{h}_2}\right)^{\frac{\mathfrak{n}}{2}}||\ell_0 - \ell_1||.
\end{aligned}$$

Letting $\mathfrak{n} \to \infty$, we get $\left(\frac{\mathfrak{h}_1}{1-\mathfrak{h}_2}\right)^{\frac{\mathfrak{n}}{2}} \to 0$ as $\mathfrak{h}_1 + \mathfrak{h}_2 < 1$. This means $||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}|| \to 0$.

Now, we have to establish that $\{\ell_\mathfrak{n}\}$ is Cauchy.
For $\mathfrak{m} > \mathfrak{n}$,

$$\begin{aligned}
||\ell_\mathfrak{n} - \ell_\mathfrak{m}|| &\leq ||\ell_\mathfrak{n} - \ell_{\mathfrak{n}+1}|| + ||\ell_{\mathfrak{n}+1} - \ell_{\mathfrak{n}+2}|| + \cdots + ||\ell_{\mathfrak{m}-1} - \ell_\mathfrak{m}|| \\
&< \left(\frac{\mathfrak{h}_1}{1-\mathfrak{h}_2}\right)^{\frac{\mathfrak{n}}{2}}||\ell_0 - \ell_1|| + \left(\frac{\mathfrak{h}_1}{1-\mathfrak{h}_2}\right)^{\frac{\mathfrak{n}+1}{2}}||\ell_0 - \ell_1|| + \cdots + \left(\frac{\mathfrak{h}_1}{1-\mathfrak{h}_2}\right)^{\frac{\mathfrak{m}-1}{2}}||\ell_0 - \ell_1||.
\end{aligned}$$

Letting $\mathfrak{n}, \mathfrak{m} \to \infty$, $||\ell_\mathfrak{n} - \ell_\mathfrak{m}|| \to 0$. Thus, $\{\ell_\mathfrak{n}\}$ is Cauchy.

By completeness of $\mathscr{H}$, there subsists $\ell_* \in \mathscr{H}$ such that $\ell_\mathfrak{n} \to \ell_*$.
From the continuity of $\mathscr{S}$, we have $\mathscr{S}(\ell_\mathfrak{n}) \to \mathscr{S}(\ell_*)$. Now,

$$\mathscr{S}(\ell_*) = \mathscr{S}(\lim_{\mathfrak{n}\to\infty} \ell_\mathfrak{n}) = \lim_{\mathfrak{n}\to\infty} \mathscr{S}(\ell_\mathfrak{n}) = \lim_{\mathfrak{n}\to\infty} \ell_{\mathfrak{n}+1} = \ell_*.$$

So, ℓ_* is an invariant point of $\mathscr{S}$.

Assuming $\ell' \in \mathscr{H}$ be another fixed point of $\mathscr{S}$. So,

$$\begin{aligned}\tau(||\ell_* - \ell'||^2) &= \tau\Big(||\mathscr{S}(\ell_*) - \mathscr{S}(\ell')||^2\Big)\\ &\leq \chi(\ell_*, \ell')\tau\Big(||\mathscr{S}(\ell_*) - \mathscr{S}(\ell')||^2\Big)\\ &\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell_* - \mathscr{S}(\ell_*)||^2 + \mathfrak{h}_2||\ell' - \mathscr{S}(\ell')||^2)\Big)\\ &\quad - \frac{1}{\delta}\zeta\Big(\tau(||\ell_* - \mathscr{S}(\ell')||^2 + ||\ell' - \mathscr{S}(\ell_*)||^2)\Big)\\ &= \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell_* - \ell_*||^2 + \mathfrak{h}_2||\ell' - \ell'||^2)\Big)\\ &\quad - \frac{1}{\delta}\zeta\Big(\tau(||\ell_* - \ell'||^2 + ||\ell' - \ell_*||^2)\Big)\\ &= -\frac{1}{\delta}\zeta\Big(\tau(||\ell_* - \ell'||^2 + ||\ell' - \ell_*||^2)\Big),\end{aligned}$$

which provides a contradiction to the hypothesis.
This means $\ell_* = \ell'$. □

Theorem 3.2. *Assume that $\mathscr{H}$ is a Hilbert space and $\mathscr{S} : \mathscr{H} \to \mathscr{H}$ be a continuous map so that for any $\mathfrak{a} \in \mathbf{R}^+$ and there exist $\mathfrak{h}_1, \mathfrak{h}_2 \in \left(0, \frac{1}{4}\right)$ for which $0 < \mathfrak{h}_1 + \mathfrak{h}_2 < \frac{1}{4}$ satisfying,*

$$\begin{aligned}\chi(\ell, \mathfrak{s})\tau\Big(||\mathscr{S}(\ell) - \mathscr{S}(\mathfrak{s})||^2\Big) \leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell - \mathscr{S}(\mathfrak{s})||^2 + \mathfrak{h}_2||\mathfrak{s} - \mathscr{S}(\ell)||^2)\Big)\\ - \frac{1}{\delta}\zeta\Big(\tau(||\ell - \mathscr{S}(\ell)||^2 + ||\mathfrak{s} - \mathscr{S}(\mathfrak{s})||^2)\Big),\end{aligned}$$

for all $\delta \geq 2$ and $\ell, \mathfrak{s} \in \mathscr{H}$. In this case, $\mathscr{S}$ admits a unique fixed point.

Proof. Suppose $\ell_1 = \mathscr{S}(\ell_0), \ell_2 = \mathscr{S}(\ell_1) = \mathscr{S}^2(\ell_0), \cdots, \ell_\mathfrak{n} = \mathscr{S}(\ell_{\mathfrak{n}-1}) = \mathscr{S}^\mathfrak{n}(\ell_0)$ for all $\mathfrak{n} \in \mathbb{N}$. Again, let us presume $\xi(\ell_0, \ell_1) \geq 1$ and since $||\ell_1 - \mathscr{S}(\ell_0)|| = ||\ell_2 - \mathscr{S}(\ell_1)|| = 0$. Now, by the Definition 3.1 we can obtain, $\xi(\ell_1, \ell_2) \geq 1$.

By mathematical induction, we get $\xi(\ell_{\mathfrak{n}}, \ell_{\mathfrak{n}+1}) \geq 1, \; \forall\, \mathfrak{n} \in \mathbb{N}$. Also,

$$\begin{aligned}
\tau(||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||^2) &= \tau\Big(||\mathscr{S}(\ell_{\mathfrak{n}-1}) - \mathscr{S}(\ell_{\mathfrak{n}})||^2\Big) \\
&\leq \chi(\ell_{\mathfrak{n}-1}, \ell_{\mathfrak{n}})\tau\Big(||\mathscr{S}(\ell_{\mathfrak{n}-1}) - \mathscr{S}(\ell_{\mathfrak{n}})||^2\Big) \\
&\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \mathscr{S}(\ell_{\mathfrak{n}})||^2 + \mathfrak{h}_2||\ell_{\mathfrak{n}} - \mathscr{S}(\ell_{\mathfrak{n}-1})||^2)\Big) \\
&\quad - \frac{1}{\delta}\zeta\Big(\tau(||\ell_{\mathfrak{n}-1} - \mathscr{S}(\ell_{\mathfrak{n}-1})||^2 + ||\ell_{\mathfrak{n}} - \mathscr{S}(\ell_{\mathfrak{n}})||^2)\Big) \\
&\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||^2 + \mathfrak{h}_2||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}}||^2)\Big) \\
&< \eta\Big(\tau(\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||^2)\Big) \\
&< \tau\Big(\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||^2\Big) \\
\Rightarrow ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||^2 &< \mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||^2 \\
\Rightarrow ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| &< \sqrt{\mathfrak{h}_1}||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}|| \\
&\leq \sqrt{\mathfrak{h}_1}\Big(||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}|| + ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||\Big) \\
\Rightarrow (1-\sqrt{\mathfrak{h}_1})||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| &< \sqrt{\mathfrak{h}_1}||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}|| \\
\Rightarrow ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| &< \frac{\sqrt{\mathfrak{h}_1}}{1-\sqrt{\mathfrak{h}_1}}||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}|| \\
&\vdots \\
&< \left(\frac{\sqrt{\mathfrak{h}_1}}{1-\sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{n}}||\ell_0 - \ell_1||.
\end{aligned}$$

Letting $\mathfrak{n} \to \infty$, $\left(\frac{\sqrt{\mathfrak{h}_1}}{1-\sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{n}} \to 0$ as $\mathfrak{h}_1 + \mathfrak{h}_2 < \frac{1}{4}$.
i.e. $||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| \to 0$.

Now, we shall establish that $\{\ell_{\mathfrak{n}}\}$ is Cauchy.
For $\mathfrak{m} > \mathfrak{n}$,

$$\begin{aligned}
||\ell_{\mathfrak{n}} - \ell_{\mathfrak{m}}|| &\leq ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| + ||\ell_{\mathfrak{n}+1} - \ell_{\mathfrak{n}+2}|| + \cdots + ||\ell_{\mathfrak{m}-1} - \ell_{\mathfrak{m}}|| \\
&< \left(\frac{\sqrt{\mathfrak{h}_1}}{1-\sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{n}}||\ell_0 - \ell_1|| + \left(\frac{\sqrt{\mathfrak{h}_1}}{1-\sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{n}+1}||\ell_0 - \ell_1|| + \cdots + \left(\frac{\sqrt{\mathfrak{h}_1}}{1-\sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{m}-1}||\ell_0 - \ell_1||.
\end{aligned}$$

Letting $\mathfrak{n}, \mathfrak{m} \to \infty$ we get $||\ell_{\mathfrak{n}} - \ell_{\mathfrak{m}}|| \to 0$.
Thus, $\{\ell_{\mathfrak{n}}\}$ is Cauchy.
By completeness of $\mathscr{H}$, there subsists $\ell_* \in \mathscr{H}$ such that $\ell_{\mathfrak{n}} \to \ell_*$.
From the continuity of $\mathscr{S}$, we have $\mathscr{S}(\ell_{\mathfrak{n}}) \to \mathscr{S}(\ell_*)$. Thus,

$$\mathscr{S}(\ell_*) = \mathscr{S}(\lim_{\mathfrak{n}\to\infty} \ell_{\mathfrak{n}}) = \lim_{\mathfrak{n}\to\infty} \mathscr{S}(\ell_{\mathfrak{n}}) = \lim_{\mathfrak{n}\to\infty} \ell_{\mathfrak{n}+1} = \ell_*.$$

So, ℓ_* is an invariant point of $\mathscr{S}$.

Assuming $\ell^{'} \in \mathscr{H}$ be another fixed point of $\mathscr{S}$. So,

$$\begin{aligned}
\tau(||\ell_* - \ell^{'}||^2) &= \tau\Big(||\mathscr{S}(\ell_*) - \mathscr{S}(\ell^{'})||^2\Big) \\
&\leq \chi(\ell_*, \ell^{'})\tau\Big(||\mathscr{S}(\ell_*) - \mathscr{S}(\ell^{'})||^2\Big) \\
&\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell_* - \mathscr{S}(\ell^{'})||^2 + \mathfrak{h}_2||\ell^{'} - \mathscr{S}(\ell_*)||^2)\Big) \\
&\quad - \frac{1}{\delta}\zeta\Big(\tau(||\ell_* - \mathscr{S}(\ell_*)||^2 + ||\ell^{'} - \mathscr{S}(\ell^{'})||^2)\Big) \\
&= \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell_* - \ell^{'}||^2 + \mathfrak{h}_2||\ell^{'} - \ell_*||^2)\Big) \\
&\quad - \frac{1}{\delta}\zeta\Big(\tau(||\ell_* - \ell_*||^2 + ||\ell^{'} - \ell^{'}||^2)\Big) \\
&= \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau((\mathfrak{h}_1 + \mathfrak{h}_2)||\ell_* - \ell^{'}||^2)\Big) \\
&< \eta\Big(\tau((\mathfrak{h}_1 + \mathfrak{h}_2)||\ell_* - \ell^{'}||^2)\Big) \\
&< \tau\Big((\mathfrak{h}_1 + \mathfrak{h}_2)||\ell_* - \ell^{'}||^2\Big), \\
\Rightarrow ||\ell_* - \ell^{'}||^2 &< (\mathfrak{h}_1 + \mathfrak{h}_2)||\ell_* - \ell^{'}||^2 \\
&< ||\ell_* - \ell^{'}||^2,
\end{aligned}$$

which provides a contradiction to the hypothesis.
This gives $\ell_* = \ell^{'}$. □

Theorem 3.3. *Consider a Hilbert space $\mathscr{H}$ and a continuous function $\mathscr{S} : \mathscr{H} \to \mathscr{H}$ such that for some $\mathfrak{a} \in (0, \infty),\ \delta > 1$ and there exist $\mathfrak{h}_1, \mathfrak{h}_2 \in \left(0, \frac{1}{4}\right)$ for which $0 < \mathfrak{h}_1 + \mathfrak{h}_2 < \frac{1}{4}$ satisfying,*

$$\chi(\ell, \mathfrak{s})\tau\Big(||\mathscr{S}(\ell) - \mathscr{S}(\mathfrak{s})||^2\Big) \leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\left(\tau\left(\frac{\mathfrak{h}_1||\ell - \mathscr{S}(\mathfrak{s})||^2 + \mathfrak{h}_2||\mathfrak{s} - \mathscr{S}(\ell)||^2}{1 + ||\ell - \mathscr{S}(\ell)||^2 + ||\mathfrak{s} - \mathscr{S}(\mathfrak{s})||^2}\right)\right) - \delta \times \zeta(\tau(||\ell - \mathfrak{s}||^2)),$$

for any $\ell, \mathfrak{s} \in \mathscr{H}$. Then $\mathscr{S}$ possesses a singular invariant point within $\mathscr{H}$.

Proof. Suppose $\ell_1 = \mathscr{S}(\ell_0), \ell_2 = \mathscr{S}(\ell_1) = \mathscr{S}^2(\ell_0), \cdots, \ell_{\mathfrak{n}} = \mathscr{S}(\ell_{\mathfrak{n}-1}) = \mathscr{S}^{\mathfrak{n}}(\ell_0)$ for all $\mathfrak{n} \in \mathbb{N}$. Again, we assume $\xi(\ell_0, \ell_1) \geq 1$ and since $||\ell_1 - \mathscr{S}(\ell_0)|| = ||\ell_2 - \mathscr{S}(\ell_1)|| = 0$. Now, by the Definition 3.1, we can obtain, $\xi(\ell_1, \ell_2) \geq 1$.

By mathematical induction, we obtain $\xi(\ell_{\mathfrak{n}}, \ell_{\mathfrak{n}+1}) \geq 1, \ \forall \ \mathfrak{n} \in \mathbb{N}$. Also,

$$\begin{aligned}
\tau\Big(||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||^2\Big) &= \tau\Big(||\mathscr{S}(\ell_{\mathfrak{n}-1}) - \mathscr{S}(\ell_{\mathfrak{n}})||^2\Big)\\
&\leq \chi(\ell_{\mathfrak{n}-1}, \ell_{\mathfrak{n}})\tau\Big(||\mathscr{S}(\ell_{\mathfrak{n}-1}) - \mathscr{S}(\ell_{\mathfrak{n}})||^2\Big)\\
&\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\left(\tau\left(\frac{\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \mathscr{S}(\ell_{\mathfrak{n}})||^2 + \mathfrak{h}_2||\ell_{\mathfrak{n}} - \mathscr{S}(\ell_{\mathfrak{n}-1})||^2}{1 + ||\ell_{\mathfrak{n}-1} - \mathscr{S}(\ell_{\mathfrak{n}-1})||^2 + ||\ell_{\mathfrak{n}} - \mathscr{S}(\ell_{\mathfrak{n}})||^2}\right)\right)\\
&\quad - \delta \times \zeta(\tau(||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}||^2))\\
&\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\left(\tau\left(\frac{\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||^2 + \mathfrak{h}_2||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}}||^2}{1 + ||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}||^2 + ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||^2}\right)\right)\\
&< \eta\left(\tau\left(\frac{\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||^2}{1 + ||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}||^2 + ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||^2}\right)\right)\\
&< \tau\left(\frac{\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||^2}{1 + ||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}||^2 + ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||^2}\right)\\
\Rightarrow ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||^2 &< \frac{\mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||^2}{1 + ||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}||^2 + ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||^2}\\
&\leq \mathfrak{h}_1||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||^2\\
\Rightarrow ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| &< \sqrt{\mathfrak{h}_1}||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}+1}||\\
&\leq \sqrt{\mathfrak{h}_1}\Big(||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}|| + ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}||\Big)\\
\Rightarrow (1 - \sqrt{\mathfrak{h}_1})||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| &< \sqrt{\mathfrak{h}_1}||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}||\\
\Rightarrow ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| &< \frac{\sqrt{\mathfrak{h}_1}}{1 - \sqrt{\mathfrak{h}_1}}||\ell_{\mathfrak{n}-1} - \ell_{\mathfrak{n}}||\\
&\ \ \vdots\\
&< \left(\frac{\sqrt{\mathfrak{h}_1}}{1 - \sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{n}}||\ell_0 - \ell_1||.
\end{aligned}$$

Letting $\mathfrak{n} \to \infty$, we get $\left(\frac{\sqrt{\mathfrak{h}_1}}{1 - \sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{n}} \to 0$ as $\mathfrak{h}_1 + \mathfrak{h}_2 < \frac{1}{4}$. This means $||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| \to 0$.
Now, we have to establish that $\{\ell_{\mathfrak{n}}\}$ is Cauchy.
For $\mathfrak{m} > \mathfrak{n}$,

$$\begin{aligned}
||\ell_{\mathfrak{n}} - \ell_{\mathfrak{m}}|| &\leq ||\ell_{\mathfrak{n}} - \ell_{\mathfrak{n}+1}|| + ||\ell_{\mathfrak{n}+1} - \ell_{\mathfrak{n}+2}|| + \cdots + ||\ell_{\mathfrak{m}-1} - \ell_{\mathfrak{m}}||\\
&< \left(\frac{\sqrt{\mathfrak{h}_1}}{1 - \sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{n}}||\ell_0 - \ell_1|| + \left(\frac{\sqrt{\mathfrak{h}_1}}{1 - \sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{n}+1}||\ell_0 - \ell_1|| + \cdots + \left(\frac{\sqrt{\mathfrak{h}_1}}{1 - \sqrt{\mathfrak{h}_1}}\right)^{\mathfrak{m}-1}||\ell_0 - \ell_1||.
\end{aligned}$$

Letting $\mathfrak{n}, \mathfrak{m} \to \infty$, $||\ell_{\mathfrak{n}} - \ell_{\mathfrak{m}}|| \to 0$.
Thus, $\{\ell_{\mathfrak{n}}\}$ is Cauchy. By completeness of $\mathscr{H}$, there subsists $\ell_* \in \mathscr{H}$ such that $\ell_{\mathfrak{n}} \to \ell_*$.
From the continuity of $\mathscr{S}$, we have $\mathscr{S}(\ell_{\mathfrak{n}}) \to \mathscr{S}(\ell_*)$. Now,

$$\mathscr{S}(\ell_*) = \mathscr{S}(\lim_{\mathfrak{n}\to\infty} \ell_{\mathfrak{n}}) = \lim_{\mathfrak{n}\to\infty} \mathscr{S}(\ell_{\mathfrak{n}}) = \lim_{\mathfrak{n}\to\infty} \ell_{\mathfrak{n}+1} = \ell_*.$$

So, ℓ_* is an invariant point of $\mathscr{S}$.

Assuming $\ell^{'} \in \mathscr{H}$ be another fixed point of $\mathscr{S}$. Also,

$$\begin{aligned}
\tau(||\ell_* - \ell^{'}||^2) &= \tau\Big(||\mathscr{S}(\ell_*) - \mathscr{S}(\ell^{'})||^2\Big) \\
&\leq \chi(\ell_*, \ell^{'})\tau\Big(||\mathscr{S}(\ell_*) - \mathscr{S}(\ell^{'})||^2\Big) \\
&\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\left(\tau\left(\frac{\mathfrak{h}_1||\ell_* - \mathscr{S}(\ell^{'})||^2 + \mathfrak{h}_2||\ell^{'} - \mathscr{S}(\ell_*)||^2}{1 + ||\ell_* - \mathscr{S}(\ell_*)||^2 + ||\ell' - \mathscr{S}(\ell')||^2}\right)\right) \\
&- \delta \times \zeta(\tau(||\ell_* - \ell^{'}||^2)) \\
&\leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell_* - \ell^{'}||^2 + \mathfrak{h}_2||\ell^{'} - \ell_*||^2)\Big) \\
&= \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau((\mathfrak{h}_1 + \mathfrak{h}_2)||\ell_* - \ell^{'}||^2)\Big) \\
&< \eta\Big(\tau((\mathfrak{h}_1 + \mathfrak{h}_2)||\ell_* - \ell^{'}||^2)\Big) \\
&< \tau\Big((\mathfrak{h}_1 + \mathfrak{h}_2)||\ell_* - \ell^{'}||^2\Big), \\
\Rightarrow ||\ell_* - \ell^{'}||^2 &< (\mathfrak{h}_1 + \mathfrak{h}_2)||\ell_* - \ell^{'}||^2 \\
&< ||\ell_* - \ell^{'}||^2,
\end{aligned}$$

which is a contradiction. Hence, $\ell_* = \ell^{'}$ Therefore, $\mathscr{S}$ possesses a unique fixed point. □

4 Illustrations

Example 4.1. *Suppose $\mathscr{H} = [0, 1]$ be a Hilbert space defined with standard norm.*
We take a mapping $\mathscr{S} : \mathscr{H} \to \mathscr{H}$ such that,

$$\mathscr{S}(\ell) = \frac{3\ell}{10(2+\ell)}, \qquad \text{for } \ell \in \mathscr{H}.$$

We have, $\mathscr{S}(0) = 0$. We take, $\chi(\ell, \mathfrak{s}) = \frac{11}{10}$, for all $\ell, \mathfrak{s} \in \mathscr{H}$.
Also, $\tau, \eta, \zeta : [0, \infty) \to [0, \infty)$ defined by,

$$\begin{aligned}
\tau(\mathfrak{j}) &= \frac{\mathfrak{j}}{6}, \\
\eta(\mathfrak{j}) &= \frac{89\mathfrak{j}}{100}, \\
\zeta(\mathfrak{j}) &= \frac{\mathfrak{j}}{1000}.
\end{aligned}$$

So, $\tau \in \Gamma, \eta \in \Sigma, \zeta \in \Omega$.
Again, we take $\mathfrak{a} = 2, \delta = 2, \mathfrak{h}_1 = \frac{1}{2}, \mathfrak{h}_2 = \frac{1}{3}$.
By computing we can easily verify,

$$\begin{aligned}
\chi(\ell, \mathfrak{s})\tau\Big(||\mathscr{S}(\ell) - \mathscr{S}(\mathfrak{s})||^2\Big) \leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell - \mathscr{S}(\ell)||^2 + \mathfrak{h}_2||\mathfrak{s} - \mathscr{S}(\mathfrak{s})||^2)\Big) \\
- \frac{1}{\delta}\zeta\Big(\tau(||\ell - \mathscr{S}(\mathfrak{s})||^2 + ||\mathfrak{s} - \mathscr{S}(\ell)||^2)\Big),
\end{aligned}$$

for all $\ell, \mathfrak{s} \in \mathscr{H}$. Thus, all conditions specified in Theorem 3.1 are fulfilled and $\mathscr{S}$ possesses a unique fixed point i.e., 0.

Example 4.2. *Suppose $\mathscr{H} = [0,1]$ be a Hilbert space defined with standard norm. We take a mapping $\mathscr{S} : \mathscr{H} \to \mathscr{H}$ such that,*

$$\mathscr{S}(\ell) = \frac{\ell}{20(1+\ell)}, \quad \text{for } \ell \in \mathscr{H}.$$

We have $\mathscr{S}(0) = 0$. We take $\chi(\ell, \mathfrak{s}) = \frac{101}{100}$, for all $\ell, \mathfrak{s} \in \mathscr{H}$. Also $\tau, \eta, \zeta : [0,\infty) \to [0,\infty)$ defined by,

$$\tau(\mathfrak{j}) = \frac{\mathfrak{j}}{8},$$
$$\eta(\mathfrak{j}) = \frac{97\mathfrak{j}}{100},$$
$$\zeta(\mathfrak{j}) = \frac{3\mathfrak{j}}{1000}.$$

So, $\tau \in \Gamma, \eta \in \Sigma, \zeta \in \Omega$. Again, we take $\mathfrak{a} = 2, \delta = 2, \mathfrak{h}_1 = \frac{1}{10}, \mathfrak{h}_2 = \frac{1}{9}$. By computing we can easily verify,

$$\chi(\ell,\mathfrak{s})\tau\Big(||\mathscr{S}(\ell) - \mathscr{S}(\mathfrak{s})||^2\Big) \leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell - \mathscr{S}(\mathfrak{s})||^2 + \mathfrak{h}_2||\mathfrak{s} - \mathscr{S}(\ell)||^2)\Big) - \frac{1}{\delta}\zeta\Big(\tau(||\ell - \mathscr{S}(\ell)||^2 + ||\mathfrak{s} - \mathscr{S}(\mathfrak{s})||^2)\Big),$$

for all $\ell, \mathfrak{s} \in \mathscr{H}$. Thus, all conditions specified in Theorem 3.2 are fulfilled. Thus, $\mathscr{S}$ admits a unique fixed point i.e., 0.

Example 4.3. *Suppose $\mathscr{H} = [0,1]$ be a Hilbert space defined with standard norm. We take a mapping $\mathscr{S} : \mathscr{H} \to \mathscr{H}$ such that,*

$$\mathscr{S}(\ell) = \frac{\ell}{30(1+\ell)}, \quad \text{for } \ell \in \mathscr{H}.$$

We have $\mathscr{S}(0) = 0$. We take $\chi(\ell, \mathfrak{s}) = \frac{21}{20}$, for all $\ell, \mathfrak{s} \in \mathscr{H}$. Also $\tau, \eta, \zeta : [0,\infty) \to [0,\infty)$ defined by,

$$\tau(\mathfrak{j}) = \frac{\mathfrak{j}}{10},$$
$$\eta(\mathfrak{j}) = \frac{19\mathfrak{j}}{20},$$
$$\zeta(\mathfrak{j}) = \frac{5\mathfrak{j}}{1051}.$$

So, $\tau \in \Gamma,\ \eta \in \Sigma,\ \zeta \in \Omega$. Again, we take $\mathfrak{a} = 10, \delta = \frac{11}{10}, \mathfrak{h}_1 = \frac{1}{10}, \mathfrak{h}_2 = \frac{1}{9}$. By computing we can easily verify,

$$\chi(\ell,\mathfrak{s})\tau\Big(||\mathscr{S}(\ell) - \mathscr{S}(\mathfrak{s})||^2\Big) \leq \frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\left(\tau\left(\frac{\mathfrak{h}_1||\ell - \mathscr{S}(\mathfrak{s})||^2 + \mathfrak{h}_2||\mathfrak{s} - \mathscr{S}(\ell)||^2}{1 + ||\ell - \mathscr{S}(\ell)||^2 + ||\mathfrak{s} - \mathscr{S}(\mathfrak{s})||^2}\right)\right) - \delta \times \zeta\Big(\tau(||\ell - \mathfrak{s}||^2)\Big),$$

for all $\ell, \mathfrak{s} \in \mathscr{H}$. Thus, all the conditions specified in the Theorem 3.3 are fulfilled. Therefore, $\mathscr{S}$ admits a unique fixed point i.e., 0.

5 Consequences

Corollary 5.1. *Consider a Hilbert space $\mathscr{H}$ and a continuous function $\mathscr{S}:\mathscr{H}\to\mathscr{H}$ such that for some $\mathfrak{a}\in(0,\infty)$ and there exist $\mathfrak{h}_1,\mathfrak{h}_2\in(0,1)$ for which $0<\mathfrak{h}_1+\mathfrak{h}_2<1$ satisfying,*

$$\begin{aligned}\tau\Big(||\mathscr{S}(\ell)-\mathscr{S}(\mathfrak{s})||^2\Big)\leq&\frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell-\mathscr{S}(\ell)||^2+\mathfrak{h}_2||\mathfrak{s}-\mathscr{S}(\mathfrak{s})||^2)\Big)\\&-\frac{1}{\delta}\zeta\Big(\tau(||\ell-\mathscr{S}(\mathfrak{s})||^2+||\mathfrak{s}-\mathscr{S}(\ell)||^2)\Big),\end{aligned}$$

for all $\delta\geq2$ and $\ell,\mathfrak{s}\in\mathscr{H}$. Then, $\mathscr{S}$ has a unique invariant point in $\mathscr{H}$.

Proof. Let us define $\chi:\mathscr{H}\times\mathscr{H}\to[0,\infty)$ such that $\chi(\ell,\mathfrak{s})=1,\ \forall\ \ell,\mathfrak{s}\in\mathscr{H}$. Therefore, all the assertions of the Theorem 3.1 hold and this establishes uniqueness of fixed point. ☐

Corollary 5.2. *Consider a Hilbert space $\mathscr{H}$ and a continuous function $\mathscr{S}:\mathscr{H}\to\mathscr{H}$ such that for some $\mathfrak{a}\in(0,\infty)$ and there exist $\mathfrak{h}_1,\mathfrak{h}_2\in\left(0,\frac{1}{4}\right)$ for which $0<\mathfrak{h}_1+\mathfrak{h}_2<\frac{1}{4}$ satisfying,*

$$\begin{aligned}\tau\Big(||\mathscr{S}(\ell)-\mathscr{S}(\mathfrak{s})||^2\Big)\leq&\frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\Big(\tau(\mathfrak{h}_1||\ell-\mathscr{S}(\mathfrak{s})||^2+\mathfrak{h}_2||\mathfrak{s}-\mathscr{S}(\ell)||^2)\Big)\\&-\frac{1}{\delta}\zeta\Big(\tau(||\ell-\mathscr{S}(\ell)||^2+||\mathfrak{s}-\mathscr{S}(\mathfrak{s})||^2)\Big),\end{aligned}$$

for all $\delta\geq2$ and $\ell,\mathfrak{s}\in\mathscr{H}$. Then, $\mathscr{S}$ possesses a unique fixed point in $\mathscr{H}$.

Proof. We define $\chi:\mathscr{H}\times\mathscr{H}\to[0,\infty)$ such that $\chi(\ell,\mathfrak{s})=1,\ \forall\ \ell,\mathfrak{s}\in\mathscr{H}$. Therefore, all the assertions of the Theorem 3.2 hold and this establishes uniqueness of fixed point. ☐

Corollary 5.3. *Consider a Hilbert space $\mathscr{H}$ and a continuous function $\mathscr{S}:\mathscr{H}\to\mathscr{H}$ such that for some $\mathfrak{a}\in(0,\infty),\delta>1$ and there exist $\mathfrak{h}_1,\mathfrak{h}_2\in\left(0,\frac{1}{4}\right)$ for which $0<\mathfrak{h}_1+\mathfrak{h}_2<\frac{1}{4}$ satisfying,*

$$\tau\Big(||\mathscr{S}(\ell)-\mathscr{S}(\mathfrak{s})||^2\Big)\leq\frac{\mathfrak{a}}{1+\mathfrak{a}}\eta\left(\tau\left(\frac{\mathfrak{h}_1||\ell-\mathscr{S}(\mathfrak{s})||^2+\mathfrak{h}_2||\mathfrak{s}-\mathscr{S}(\ell)||^2}{1+||\ell-\mathscr{S}(\ell)||^2+||\mathfrak{s}-\mathscr{S}(\mathfrak{s})||^2}\right)\right)-\delta\times\zeta\Big(\tau(||\ell-\mathfrak{s}||^2)\Big),$$

for any $\ell,\mathfrak{s}\in\mathscr{H}$. Then, $\mathscr{S}$ possesses a unique fixed point in $\mathscr{H}$.

Proof. We define $\chi:\mathscr{H}\times\mathscr{H}\to[0,\infty)$ satisfying $\chi(\ell,\mathfrak{s})=1,\ \forall\ \ell,\mathfrak{s}\in\mathscr{H}$. Therefore, all the assertions of the Theorem 3.3 hold and this establishes uniqueness of fixed point. ☐

6 Conclusion

In this research, fixed point results for distinct mappings within the context of Hilbert spaces are explored, guided by the principles of functional analysis. By introducing new rational contractions, significant fixed point theorems are not only established but also illustrated these theoretical findings with concrete examples. These examples serve to clarify and validate the practical applicability of our theoretical results, thereby enhancing their comprehensibility and demonstrating their relevance to real-world scenarios. The corollaries derived from our primary results provide specialized instances that further underscore the versatility and robustness of the fixed point theorems under the newly proposed rational contractions. These corollaries facilitate a deeper understanding by highlighting particular cases where the general results can be applied, thereby broadening the scope of potential applications. Through meticulous iterative processes, we have rigorously validated the conditions under which these rational contractions guarantee the existence of fixed points. This methodological rigor ensures the reliability and accuracy of our findings, contributing meaningfully to the existing body of knowledge in the field of functional analysis and fixed point theory. Our work opens up new avenues for future research, particularly in extending these results to other types of spaces or exploring different kinds of contractions. Additionally, the insights gained from this study may have implications for various applied fields, such as optimization, computational mathematics, and the analysis of dynamic systems.

While this study forms a solid foundation for understanding fixed points under rational contractions, several promising avenues for future research emerge such as investigating the applicability of rational contractions in diverse spaces such as Banach spaces and normed spaces could broaden the scope and applicability of our findings, extending our results to higher-dimensional spaces and complex systems can unveil new theoretical insights and practical applications, developing efficient algorithms based on our methods and principles could enhance computational approaches in scientific and engineering applications, applying established fixed point results to analyze dynamic systems in physics, biology, and economics could provide new perspectives on system behaviors and stability, exploring further generalizations of rational contraction conditions may lead to more robust theorems and expanded theoretical frameworks, investigating applications in optimization, control theory, and machine learning can bridge theoretical research with practical implementation. By pursuing these future research directions, we can build upon this study foundation, uncovering new applications and advancing our understanding of fixed point theory across diverse disciplines.

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